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and the vertices another circle; and so on *ad infinitum*. Show that the ratio of the sum of the areas of all the circles to the area of the triangle is

$$\frac{\Sigma}{\Delta} = \frac{\pi}{4}\frac{\Delta}{S^2}\left[\frac{1}{\sin\frac{1}{2}A} + \frac{1}{\sin\frac{1}{2}B} + \frac{1}{\sin\frac{1}{2}C} - 2 + \sin\tfrac{1}{2}A + \sin\tfrac{1}{2}B + \sin\tfrac{1}{2}C\right].$$

2897. Proposed by PAUL CAPRON, U. S. Naval Academy.

Discuss the conditions under which the angles made by two circles on a sphere have the same measures as the distances between their poles.

2898. Proposed by J. W. CLAWSON, Ursinus College.

Four straight lines determine four triangles. It is well known that the circumcenters of these triangles lie on a circle and that the circumcircles intersect this circle in a point, called the Wallace point. It is also well known that the orthocenters of the four triangles lie on a straight line, which is perpendicular to the line on which lie the middle points of the three diagonals of the quadrilateral determined by the four given straight lines.[1]

Prove that the centroids of the four triangles lie on a parabola whose axis is parallel to the mid-diagonal line; and that the distance from the Wallace point to the mid-diagonal line is two-thirds of the distance from the Wallace point to the axis of the parabola.

2899. Proposed by NORMAN ANNING, University of Michigan.

A, B, C, and P are any four coplanar points. P describes a sextant about A when the line AP turns about A through $+60°$. Show that P moves in a closed curve when it describes sextants in succession *either* about $A, B, A, B, A, \cdots$ *or* about $A, B, C, A, B, C, \cdots$.

PROBLEMS—NOTES.

13. Professor C. N. MILLS, of Heidelberg University, Tiffin, Ohio, proposed the following problem: "A, D and C are telegraph poles at equal intervals by the side of a railroad; t_1 and t_2 are the tangents of the angles which AD and DC subtend at any point B on the road; t is the tangent of the angle which DB makes with DC. Show that $2/t = 1/t_1 - 1/t_2$." This relation may be written down at once from the familiar results[2] that $1/t = (\cot A - \cot C)/2$ where $A = \angle BAC$, $C = \angle BCA$; $1/t_1 = 2\cot B + \cot A$, where $B = \angle ABC$; and $1/t_2 = 2\cot B + \cot C$. If s_1, s_2 and s are the sines of the angles whose tangents are t_1, t_2 and t, respectively, it may be readily shown[3] that $s_1 = (a\sin B)/d$, $s_2 = (b\sin C)/d$, $s = (a\sin C)/d$, where a, b, c have the usual significance in connection with the triangle ABC, and $d = \sqrt{2a^2 + 2c^2 - b^2}$. ARC.

14. Professor I. A. BARNETT, of the University of Saskatchewan, proposed the problem: "Prove that

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \left(\frac{1}{2}\right)^7."$$

This result is given as example 13 on page 319 of W. E. Johnson, *Treatise on Trigonometry*, London, 1889, and as example 5 on page 119 of E. W. Hobson, *A Treatise on Plane Trigonometry*, second edition, Cambridge, 1897. The solution follows readily from the preceding theory where it is shown that

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[1] Steiner, *Annales de Mathématiques Pures et Appliquées*, April, 1828.—EDITOR.

[2] See, for example, J. Casey, *A Treatise on Plane Trigonometry*, Dublin, 1888, pp. 174–175; S. L. Loney, *Plane Trigonometry*, Cambridge, 1893, p. 246.

[3] S. L. Loney, *l.c.*, p. 240.

$$2^{\frac{1}{2}(n-1)}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\cdots\sin\frac{(n-2)\pi}{2n}=1 \qquad \text{when } n \text{ is odd,}$$

and

$$2^{\frac{1}{2}(n-1)}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\cdots\sin\frac{(n-1)\pi}{2n}=1 \qquad \text{when } n \text{ is even.}$$

ARC.

15. In the discussion of problem 2819 [*1920*, 134, 187–190] concerning the envelope of the system of circles inscribed in a triangle with a given base, *AB*, and a given vertical angle, *C*, it is remarked that, if we regard the variable triangle on one side of the base only, the locus of the centers of the system of circles is an arc of a circle through *A* and *B*, and whose center bisects the arc, on the opposite side of *AB*, of the circumscribing circle. This locus was noticed by John Turner, in *The Mathematician*, no. 5, 1749, p. 312, in the course of his answer to the problem he had proposed the previous year: One side of a triangle, together with the radii of its circumscribing and inscribed circles being given, to construct the triangle geometrically. The locus was also discussed by N. Fuss,[1] in 1794; by J. Bonnycastle,[2] in 1818; by D. Cresswell,[3] in 1819; and by Holleben and Gerwien,[4] in 1831, along with the corresponding locus for the escribed circle in *C*. These loci, and that for a second escribed circle, were called for by J. Luby[5] in 1833.

It is concerning the locus of the incenter that the philosopher Herbert Spencer wrote in his *Autobiography* (New York, 1904), volume 1, pp. 187–188:

> "When seventeen I hit on a geometrical theorem of some interest. This remained with me in the form of an empirical truth; but during the latter part of my residence in Worcester, responding to a spur from my father, I made a demonstration of it; and, now that it had reached this developed form, it was published in *The Civil Engineer and Architect's Journal* for July, 1840. It is reproduced in Appendix B. I did not know, at the time, that this theorem belongs to that division of mathematics at one time included under the name 'Descriptive Geometry,' but known in more recent days as 'The Geometry of Position'—a division which includes many marvellous truths. Perhaps the most familiar of these is the truth that if to three unequal circles anywhere placed, three pairs of tangents be drawn, the points of intersection of the tangents fall in the same straight line—a truth which I never contemplate without being struck by its beauty at the same time that it excites feelings of wonder and of awe: the fact that apparently unrelated circles should in every case be held together by this plexus of relations, seeming so utterly incomprehensible. The property of a circle which is enunciated in my own theorem, has nothing like so marvellous an aspect, but is nevertheless sufficiently remarkable."

The article containing "my own theorem,"[6] and a long-winded proof, is reproduced on pages 606–608 of his *Autobiography*, volume 1. The above quotation indicates that Spencer was innocent of knowledge of even the elementary ideas of geometry.[1] This and other instances of such ignorance are set forth by J. S. Mackay in "Herbert Spencer and mathematics," *Proceedings of the Edinburgh Mathematical Society*, 1907, volume 25, pp. 95–106. ARC.

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[1] *Nova Acta Acad. Sc. Imp. Petrop.*, vol. 10 (1792), 1797, pp. 124–125.

[2] J. Bonnycastle, *Elements of Geometry*, sixth edition, London, 1818, pp. 310–311.

[3] D. Cresswell, *A Supplement to the Elements of Euclid*, Cambridge, 1819, pp. 256–257; second ed., 1825, pp. 227–228.

[4] H. v. Holleben and P. Gerwien, *Geometrische Analysis*, vol. 1, 1831, p. 43; vol. 2, 1832, p. 53.

[5] J. Luby, *The Elements of Geometry . . . Also a variety of problems and theorems . . . with analysis*, Dublin, 1833, third part, p. 57.

[6] Discussed, as we have seen, about a hundred years before. References have been given to only a few of the numerous earlier discussions of the theorem and its extensions.

PROBLEMS—SOLUTIONS

2800 [1920, 31]. Proposed by A. M. HARDING, University of Arkansas.

If $x + y + z = xyz$, show that

$$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\cdot\frac{2y}{1-y^2}\cdot\frac{2z}{1-z^2}.$$

SOLUTION BY ARNOLD DRESDEN, University of Wisconsin.

Except for values of x and y for which $xy = 1$, the relation (1) $x + y + z = xyz$ is equivalent to $z = (x + y)/(xy - 1)$. Direct substitution of this value for z shows that the formula

$$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\cdot\frac{2y}{1-y^2}\cdot\frac{2z}{1-z^2} \tag{2}$$

holds for all values of x, y, except $x = \pm 1$; $y = \pm 1$; $xy = 1$, for which values one or the other of these equations loses meaning.

It is readily seen that an infinite number of other formulas will hold true as a consequence of (1), viz.,

$$\Sigma \tan (n \tan^{-1} x) = \Pi \tan (n \tan^{-1} x), \text{ for } n = 1, 2, \cdots, \tag{3}$$

the sum and product to be extended over x, y, and z. For since $x + y + z = xyz$, it follows that $-\tan^{-1} x - \tan^{-1} y = \tan^{-1} z$, for some determination of $\tan^{-1} z$; in virtue of this relation we have then $-n \tan^{-1} x - n \tan^{-1} y = n \tan^{-1} z$ and therefore:

$$-\tan (n \tan^{-1} z) = \tan (n \tan^{-1} x + n \tan^{-1} y) = \frac{\tan (n \tan^{-1} x) + \tan (n \tan^{-1} y)}{1 - \tan (n \tan^{-1} x)\cdot\tan (n \tan^{-1} y)};$$

i.e., $\Sigma \tan (n \tan^{-1} x) = \Pi \tan (n \tan^{-1} x)$.

It would be interesting to know the general solution of the functional equation:

$$\sum_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} f(x_i),$$

under the restriction $\Sigma x_i = \Pi x_i$.

Also solved by NORMAN ANNING, T. M. BLAKSLEE, S. M. BERG, J. A. BULLARD, H. N. CARLETON, E. S. HAMMOND, C. N. MILLS, E. J. OGLESBY, H. L. OLSON, J. L. RILEY, ARTHUR PELLETIER, H. S. UHLER, W. R. WARNE, and C. C. WYLIE.

2802 [1920, 31]. Proposed by WARREN WEAVER, California Institute of Technology.

Consider two circles, each of radius k, with centers at $(0, 0)$ and $(k', 0)$ respectively, where k' is less than k. Through the point $(k', 0)$ draw a ray making an angle θ with the positive x-axis. Call the intersection of this line with the first circle A, and with the second circle B. Extend the line through the point $(k', 0)$ in the opposite direction, and call the intersection of this extension with the first circle A', and with the second circle B'. Prove that the sum of the two segments AB and $A'B'$ is independent of k, and depends only upon k', i.e. the shift of the circles, and θ.

SOLUTION BY U. G. MITCHELL, University of Kansas.

Let the points A, B, A', B' have coördinates (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), respectively. By the conditions of the problem the points $A = (x_1, y_1)$ and $A' = (x_3, y_3)$ are on the circle $x^2 + y^2 = k^2$, the points $B = (x_2, y_2)$ and $B' = (x_4, y_4)$ are on the circle $(x - k')^2 + y^2 = k^2$, and all four of the points A, B, A', B' are on the line $y = m(x - k')$, where $m = \tan \theta$.

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[1] His boasting in the preface to the American edition, in 1876, of *Inventional Geometry: a series of problems*, by his father W. G. Spencer, will be recalled.